MATH 2028 Honours Advanced Calculus II
2024-25 Term 1
Suggested Solution to Problem Set 1

Notations: We use R to denote a rectangle in $\mathbb{R}^n$ throughout this problem set.

Problems to hand in

1. Let $f : R = [0, 1] \times [0, 1] \to \mathbb{R}$ be a bounded function defined by

$$f(x, y) := \begin{cases} 1 & \text{if } y < x, \\ 0 & \text{if } y \geq x. \end{cases}$$

Prove, using the definition, that f is integrable and find $\int_R f \, dV$.

Solution. For each $n \in \mathbb{N}$, let $\mathcal{P}_n := \{C_{i,j} := [\frac{i-1}{n}, \frac{i}{n}] \times [\frac{j-1}{n}, \frac{j}{n}] : 1 \leq i, j \leq n\}$ be a partition of R. By counting the number of rectangles in $\mathcal{P}_n$ lying inside the region $y < x$, it is easy to see that

$$L(f, \mathcal{P}_n) = \frac{(n-1)(n-2)}{2n^2} \quad \text{and} \quad U(f, \mathcal{P}_n) = \frac{(n+1)n}{2n^2}.$$

Thus,

$$\frac{1}{2} \leq \sup_{\mathcal{P}} L(f, \mathcal{P}) \leq \inf_{\mathcal{P}} U(f, \mathcal{P}) \leq \frac{1}{2}.$$

By definition, f is integrable on R and $\displaystyle\int_R f \, dV = \frac{1}{2}$. □

2. Let $f : R = [0, 1] \times [0, 1] \to \mathbb{R}$ be the function

$$f(x, y) := \begin{cases} 1/q & \text{if } x, y \in \mathbb{Q} \text{and } y = p/q \text{ where } p, q \in \mathbb{N} \text{ are coprime}, \\ 0 & \text{otherwise} \end{cases}$$

Prove, using the definition, that f is integrable and find $\int_R f \, dV$.

Solution. we know $f(x, y) \geq 0$, so $\int_R f dV \geq 0$, on the other hand , let
$X_n = \{x \in \mathbb{Q} | x = p/q, p \leq q \in \mathbb{Z}\}$ number of object of X_n will not bigger than $1 + 2 + 3 + \ldots n = n(n + 1)/2$ .then we can define $Y_n = \{y \in \mathbb{R} | \text{exist some } x \in X_n \text{ such that } |y - x| < 1/n^4\}$

$$\int_R f(x, y) dV = \int_{[0,1]\times Y_n} f dV + \int_{R\setminus[0,1]\times Y_n} f dV \leq 2 \times \frac{1}{n^4} \times \frac{n(n+1)}{2} + \int_{R\setminus[0,1]\times Y_n} \frac{1}{n+1} dV \leq \frac{n+1}{n^3} + \frac{1}{n+1} \tag{1}$$

let $n \to \infty$, we get $\int_R f \leq 0$, so $\int_R f = 0$.

□

3. Suppose $f : R \to \mathbb{R}$ is a non-negative *continuous* function such that $f(p) > 0$ at some $p \in R$. Prove that $\int_R f \, dV > 0$.

Solution. Let $\varepsilon_0 := \frac{f(p)}{2} > 0$. Since f is continuous at p, there exists $\delta > 0$ such that for all $x \in B_\delta(p) \cap R$, we have $|f(x) - f(p)| < \varepsilon_0 = \frac{f(p)}{2}$, and hence, by triangle inequality,

$$f(x) = |f(x)| > \frac{|f(p)|}{2} = \varepsilon_0.$$

Choose a rectangle R' such that $p \in R' \subseteq B_\delta(p) \cap R$.

Now, given any partition $\mathcal{P}$ of R, we have

$$\begin{aligned} U(f, \mathcal{P}) &= \sum_{K \in \mathcal{P}} \sup_{x \in K} f(x) \mathrm{Vol}(K) \\ &\geq \sum_{K \in \mathcal{P}, K \cap R' \neq \emptyset} \sup_{x \in K} f(x) \mathrm{Vol}(K) \\ &\geq \varepsilon_0 \sum_{K \in \mathcal{P}, K \cap R' \neq \emptyset} \mathrm{Vol}(K) \\ &\geq \varepsilon_0 \mathrm{Vol}(R'). \end{aligned}$$

Since f is continuous, hence integrable, we have

$$\int_R f\, dV = \inf_{\mathcal{P}} U(f, \mathcal{P}) \geq \varepsilon_0 \mathrm{Vol}(R') > 0.$$

□

4. Let $f : R \to \mathbb{R}$ be a bounded integrable function . Prove that $|f|$ also integrable on R and $|\int_R f dV| \leq \int_R |f| dV$.

Solution. easily we know $|x - y| \geq ||x| - |y||$, so

$$U(f, P) - L(f, P) = \sum_{P_i \in P} [\sup_{x \in P_i} f(x) - \inf_{x \in P_i} f(x)] Vol(P_i) \geq \sum_{P_i \in P} [\sup_{x \in P_i} |f(x)| - \inf_{x \in P_i} |f(x)|] Vol(P_i) \quad (2)$$

$= U(|f|, P) - L(|f|, P)$ so we know $|f|$ integrable , on the other hand , $-|f| \leq f \leq |f|$, $-\int_R |f| dV = \int_R -|f| dV \leq \int_R f dV \leq \int_R |f| dV$, which means $|\int_R f dV| \leq \int_R |f| dV$

□

5. Let $f : R \to \mathbb{R}$ be a bounded integrable function. Suppose p is an interior point of R at which f is continuous . Prove that

$$\lim_{\delta \to 0^+} \frac{1}{\mathrm{Vol}(B_\delta(p))} \int_{B_\delta(p)} f\, dV = f(p). \quad (3)$$

Solution. because f continue , so for all $\varepsilon > 0$ there exist $\delta > 0$ such that when $d(x, p) < \delta$ we can get $|f(x) - f(p)| < \varepsilon$, so we have

$$\frac{1}{\mathrm{Vol}(B_\delta(p))} |\int_{B_\delta(p)} f - f(p) dV| \leq \frac{1}{\mathrm{Vol}(B_\delta(p))} \int_{B_\delta(p)} |\varepsilon| dV = \frac{1}{\mathrm{Vol}(B_\delta(p))} \times B_\delta(p) \times \varepsilon = \varepsilon \quad (4)$$

so when $\delta \to 0$ we can let $\varepsilon \to 0$ then finally get

$$\lim_{\delta\to 0^+} \frac{1}{\mathrm{Vol}(B_\delta(p))} \int_{B_\delta(p)} f\, dV = f(p). \tag{5}$$

□

Suggested Exercises

1. Let $f, g : R \to \mathbb{R}$ be bounded integrable functions. Prove that $f+g$ is integrable on R and

$$\int_R (f+g)\ dV = \int_R f\ dV + \int_R g\ dV.$$

Solution. Let $\mathcal{P}_1', \mathcal{P}_2'$ be two partitions of R. Let $\mathcal{P}'$ be a common refinement of $\mathcal{P}_1', \mathcal{P}_2'$. By the properties of infimum and refinement,

$$L(f+g, \mathcal{P}') \geq L(f, \mathcal{P}') + L(g, \mathcal{P}') \geq L(f, \mathcal{P}_1') + L(g, \mathcal{P}_2').$$

So,

$$\sup_{\mathcal{P}} L(f+g, \mathcal{P}) \geq L(f, \mathcal{P}_1') + L(g, \mathcal{P}_2').$$

Since $\mathcal{P}_1', \mathcal{P}_2'$ are arbitrary, we have

$$\sup_{\mathcal{P}} L(f+g, \mathcal{P}) \geq \sup_{\mathcal{P}_1} L(f, \mathcal{P}_1) + \sup_{\mathcal{P}_2} L(g, \mathcal{P}_2). \tag{6}$$

By a similar argument, we see that

$$\inf_{\mathcal{P}} U(f+g, \mathcal{P}) \leq \inf_{\mathcal{P}_1} U(f, \mathcal{P}_1) + \inf_{\mathcal{P}_2} U(g, \mathcal{P}_2). \tag{7}$$

As f, g are bounded integrable functions on R, we have

$$\sup_{\mathcal{P}_1} L(f, \mathcal{P}_1) = \inf_{\mathcal{P}_1} U(f, \mathcal{P}_1) = \int_R f\, dV \quad \text{and} \quad \sup_{\mathcal{P}_2} L(g, \mathcal{P}_2) = \inf_{\mathcal{P}_2} U(f, \mathcal{P}_2) = \int_R g\, dV.$$

Hence, (6) and (7) imply that

$$\int_R f\, dV + \int_R g\, dV \leq \sup_{\mathcal{P}} L(f+g, \mathcal{P}) \leq \inf_{\mathcal{P}} U(f+g, \mathcal{P}) \leq \int_R f\, dV + \int_R g\, dV.$$

Therefore,

$$\sup_{\mathcal{P}} L(f+g, \mathcal{P}) = \inf_{\mathcal{P}} U(f+g, \mathcal{P}) = \int_R f\, dV + \int_R g\, dV.$$

By definition, $f+g$ is integrable on R and

$$\int_R (f+g)\, dV = \int_R f\, dV + \int_R g\, dV.$$

□

2. Let $f : R \to \mathbb{R}$ be a bounded integrable function defined on a rectangle $R \subset \mathbb{R}^n$. Suppose $g : R \to \mathbb{R}$ is a bounded function such that $g(x) = f(x)$ except for finitely many $x \in R$. Show that g is integrable and $\int_R g\, dV = \int_R f\, dV$.

   Solution. Let $\mathcal{P}$ be a partition of R. Since a point in R is contained in at most 2^n rectangles in $\mathcal{P}$, the upper sum (and lower sum) of f and g differ by at most

$$\#\{x \in R : f(x) \neq g(x)\} \cdot 2^n \cdot \sup_{x \in R} |f(x) - g(x)| \cdot \sup_{C \in \mathcal{P}} \mathrm{Vol}(C).$$

   As the partition $\mathcal{P}$ gets finer and finer, $\sup\limits_{C \in \mathcal{P}} \mathrm{Vol}(C) \to 0$. It is then straightforward to show that g is also integrable and $\int_R g\, dV = \int_R f\, dV$. □

Challenging Exercises

1. Let f be a bounded integrable function on R. Prove that for any $\epsilon > 0$, there exists some $\delta > 0$ such that whenever $\mathcal{P}$ is a partition of R with $\mathrm{diam}(Q) < \delta$ for all $Q \in \mathcal{P}$, and $x_Q \in Q$ is any arbitrarily chosen point inside $Q \in \mathcal{P}$, we have

$$\left| \sum_{Q \in \mathcal{P}} f(x_Q)\mathrm{Vol}(Q) - \int_R f\, dV \right| < \epsilon.$$

   (The sum in the above expression is what we usually call the "Riemann sum"!)

   Solution. It suffices to note that given a partition $\mathcal{P}$ and arbitrarily chosen points $x_Q \in Q$ for each $Q \in \mathcal{P}$, we have

$$L(f, \mathcal{P}) \leq \sum_{Q \in \mathcal{P}} f(x_Q)\mathrm{Vol}(Q) \leq U(f, \mathcal{P}),$$

   and

$$L(f, \mathcal{P}) \leq \int_R f\, dV \leq U(f, \mathcal{P}).$$

   □